Problem Sheet 4

① $\Omega \subset \mathbb{R}^n$ open, bounded. $v \in C^2(\bar\Omega)$ subharmonic if
$-\Delta v(x) \leq 0 \quad \forall x \in \Omega$.

(i) <u>Show</u>: if v subharmonic, $v(x) \leq \fint_{B(x,r)} v(y)\,dy \quad \forall B(x,r) \subset\subset \Omega$.

Let $B(x,r) \subset\subset \Omega$. Then for $s \in (0,r)$ set $\varphi(s) = \fint_{\partial B(x,s)} v(y)\,dS(y)$

Then (as in lectures / tutorials / last week) we have $\varphi \in C^1(0,r)$ and

$$\varphi'(s) = \frac{1}{s^{n-1}\omega_n}\int_{B(x,s)} \underbrace{\Delta v(y)}_{\geq 0}\,dy \geq 0 \quad (v$$

$\lim_{s \downarrow 0} \varphi(s) = v(x)$. Also, for all $s \in (0,r)$ we have

$$0 \leq \int_0^s \underbrace{\varphi'(t)}_{\geq 0}\,dt = \varphi(s) - \varphi(0) = \varphi(s) - v(x)$$

So $v(x) \leq \varphi(s) \quad \forall s \in (0,r)$. Now note that

$$\int_{B(x,r)} v(y)\,dy \overset{\text{polar coords}}{=} \int_0^r \int_{\partial B(x,s)} v(y)\,dS(y)\,ds$$

$$= \int_0^r s^{n-1}\omega_n\,\varphi(s)\,ds$$

$$\geq \int_0^r s^{n-1}\omega_n\,v(x)\,ds = v(x)\int_0^r \underbrace{s^{n-1}\omega_n}_{=\mathcal{H}^{n-1}(\partial B(x,s))}\,ds$$

$$= v(x)\,\alpha_n r^n.$$

So $v(x) \leq \fint_{B(x,r)} v(y)\,dy$

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(ii) <u>Claim</u>: If v subharmonic, $\max_{x \in \bar{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x)$

<u>Pf.</u> Write $M = \max_{x \in \bar{\Omega}} v(x)$. Suppose $\exists x_0 \in \Omega$ s.t. $v(x_0) = M$.

We want to show that this implies v is constant so claim holds still.

Ω open, so $\exists r > 0$ s.t. $B(x_0, r) \subset\subset \Omega$. Then, by (i)

$$M = v(x_0) \leq \fint_{B(x_0,r)} v(y)\, dy \leq M$$

Hence $v(y) = M \quad \forall y \in B(x_0, r)$

i.e ~~(v is continuous)~~ $v^{-1}(\{M\})$ is open.

But since v is cont, $\{M\}$ closed in $\mathbb{R}$, $v^{-1}(\{M\})$ is also closed! Hence, if Ω is connected, $v(y) = M \quad \forall y \in \bar{\Omega}$.

Otherwise $v = $ constant on each connected component of Ω.

So still we have $\max_{x \in \bar{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x)$.

(iii) ~~For~~ <u>Recall</u>: for $B = B(x, r) \subset\subset \Omega$

$$\cancel{H_B v(y) := \begin{cases} \fint_{\partial B} K_B(z, y)\, v(y)\, dS(y) & y \in B \\ v(y) & y \notin B \end{cases}}$$

$$K_B(z, y) = \frac{1}{r\omega_n} \frac{r^2 - |z - x|^2}{|z - y|^n} \quad \text{Poisson kernel.}$$

<u>lectures</u>: $H_B v$ harmonic on B and $H_B v = v$ on ∂B.

<u>~~Show~~</u>:

$$H_B v(y) := \begin{cases} \int_{\partial B} K_B(y, z)\, v(z)\, dS(z) & y \in B \\ v(y) & y \notin B \end{cases}$$

$$K_B(y, z) = \frac{r^2 - |y - x|^2}{r\omega_n |z - y|^n} \quad \text{Poisson kernel}$$

Show: $v(y) \leq H_B(y) \quad \forall y \in B$. for v subharmonic.

Note $-\Delta(v - H_B v) = -\Delta v + \Delta H_B v = -\Delta v$. on B.

So $v - H_B v$ is subharmonic too (on B). So by (ii)

So $\max_{y \in \bar{B}} (v(y) - H_B v(y)) = \max_{y \in \partial B} (v(y) - H_B v(y)) = 0$

($v = H_B v$ on ∂B!)

So $\forall y \in B$, $v(y) - H_B v(y) \leq 0$ ie $v(y) \leq H_B v(y)$.

(iv) $u \in C^2(\bar{\Omega})$ harmonic, $\varphi \in C^\infty(\mathbb{R})$ convex,

$v(x) := \varphi(u(x))$. $x \in \bar{\Omega}$.

Show: v sub-harmonic.

$$\frac{\partial v}{\partial x_i}(x) = \frac{\partial u}{\partial x_i}(x)\, \varphi'(u(x))$$

$$\frac{\partial^2 v}{\partial x_i^2}(x) = \frac{\partial u}{\partial x_i}(x)\, \partial/\partial x_i \left(\varphi'(u(x))\right) + \frac{\partial^2 u}{\partial x_i^2}\, \varphi'(u(x))$$

$$= \left(\frac{\partial u(x)}{\partial x_i}\right)^2 \varphi''(u(x)) + \frac{\partial^2 u}{\partial x_i^2}(x)\, \varphi'(u(x))$$

So
$$\Delta v(x) = \sum_{i=1}^n v_{x_i x_i}(x) = \sum_{i=1}^n \overbrace{(u_{x_i}(x))^2}^{\geq 0} \overbrace{\varphi''(u(x))}^{\geq 0} + \sum_{i=1}^n u_{x_i x_i}(x)\, \varphi'(u(x))$$

$$\geq \Delta u(x)\, \varphi'(u(x)) = 0.$$

So v subharmonic.

(v) $u \in C^2(\bar{\Omega})$ harmonic. Show $v(x) := |\nabla u(x)|^2$, $x \in \bar{\Omega}$ subharmonic

(7)

Pf. Note u harmonic $\Rightarrow$ u smooth on Ω (so C^3)

$$v(x) = u_{x_1}^2 + \dots + u_{x_n}^2.$$

$$v_{x_i} = 2(u_{x_1 x_i} u_{x_1} + \dots + u_{x_n x_i} u_{x_n}) = 2\sum_{j=1}^n u_{x_j x_i} \cdot u_{x_j} \quad (\bar{x})$$

$$v_{x_i x_i} = 2\sum_{j=1}^n (u_{x_j x_i} u_{x_j})_{x_i}$$

$$= 2\sum_{j=1}^n u_{x_j x_i x_i} u_{x_j} + u_{x_j x_i}^2$$

So $\Delta v = 2\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_j x_i x_i} u_{x_j} + \underbrace{(u_{x_i x_j})^2}_{\geq 0}$

$$= 2\sum_{j=1}^{n}\sum_{i=1}^{n} (u_{x_i x_i})_{x_j} u_{x_j} + (u_{x_i x_j})^2$$

$$\geq 2\sum_{j=1}^{n}\sum_{i=1}^{n} (u_{x_i x_i})_{x_j} u_{x_j}$$

So $\Delta v = 2\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_j x_i x_i} u_{x_j} + \underbrace{(u_{x_i x_j})^2}_{\geq 0}$

$$\geq 2\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_j x_i x_i} u_{x_j}$$

$$= 2\sum_{j=1}^{n} u_{x_j}\left(\sum_{i=1}^{n} u_{x_i x_i}\right)_{x_j} = 2\sum_{j=1}^{n} u_{x_j} (\Delta u)_{x_j} = 0.$$

So $-\Delta v \leq 0$.

(2) (i) For $x \in \mathbb{R}^n \setminus \{0\}$, define $\bar{x} := \frac{x}{|x|^2}$ (note $\bar{(\bar{x})} = x$)

$|\bar{x}| = \frac{1}{|x|}$.

! $D_x \bar{x} \in \mathbb{R}^{n \times n}$! if $v(x) = \bar{x} = \left(\frac{x_1}{|x|^2}, \dots, \frac{x_n}{|x|^2}\right) = (v^1 \dots v^n)$

$$Dv(x) = \begin{pmatrix} v^1_{x_1} & \dots & v^1_{x_n} \\ v^2_{x_1} & & \vdots \\ \vdots & & \vdots \\ v^n_{x_1} & & v^n_{x_n} \end{pmatrix}$$

Show $(D_x \bar{x})(D_x \bar{x})^t = |\bar{x}|^4 I$

Note $\bar{x} = (x_1, \dots, x_n)(x_1^2 + \dots + x_n^2)^{-1}$

Consider $\partial/\partial x_i$ of j^{th} component of $\bar{x}$.

$$\partial/\partial x_i \left(x_j (x_1^2 + \dots + x_n^2)^{-1} \right) = \delta_{i,j} (x_1^2 + \dots + x_n^2)^{-1} + x_j(-2x_i)(x_1^2 + \dots + x_n^2)$$

$$= \frac{\delta_{i,j}}{|x|^2} - \frac{2x_i x_j}{|x|^4} \qquad \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Write $\nabla_x \bar{x} = (a_{ij})$

and $\nabla_x \bar{x} \, (\nabla_x \bar{x})^t = (b_{ij})$ Then $b_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$

So $b_{ij} = \sum_{k=1}^{n} \left(\frac{\delta_{i,k}}{|x|^2} - \frac{2 x_i x_k}{|x|^4} \right) \left(\frac{\delta_{j,k}}{|x|^2} - \frac{2 x_j x_k}{|x|^4} \right)$

$$= \sum_{k=1}^{n} \frac{\overbrace{\delta_{i,k}\delta_{j,k}}^{=\{ 1 \ i=j, \ 0 \ i\neq j}}{|x|^4} - \frac{2\delta_{i,k} x_j x_k}{|x|^6} - \frac{2\delta_{j,k} x_i x_k}{|x|^6} + \frac{4x_k^2 x_i x_j}{|x|^8}$$

$$= \frac{\delta_{i,j}}{|x|^4} - \frac{4 x_i x_j}{|x|^6} + \frac{4 x_i x_j}{|x|^8} \underbrace{\left(\sum_{k=1}^{n} x_k^2 \right)}_{= |x|^2}$$

$$= \frac{\delta_{i,j}}{|x|^4}.$$

So $\nabla_x \bar{x} \, (\nabla_x \bar{x})^t = \frac{1}{|x|^4} I = |\bar{x}|^4 I$

(ii) $\Omega \subset \mathbb{R}^n \setminus \{0\}$ open, $n \geq 2$. Kelvin transform $\bar{u}$, Ku of u defined as

$$\bar{u}(x) := u(\bar{x}) \, |\bar{x}|^{n-2}$$
$$= u\left(\frac{x}{|x|^2}\right) |x|^{2-n}$$

Claim: u harmonic on $\Omega \Rightarrow$ so is $\bar{u}$.

Product rule for Laplace op:

$$\Delta \bar{u}(x) = \Delta \left(u(\bar{x}) \, |x|^{2-n} \right)$$
$$= u(\bar{x}) \, \Delta(|x|^{2-n}) + 2 \nabla(u(\bar{x})) \cdot \nabla(|x|^{2-n}) + |x|^{2-n} \Delta(u(\bar{x}))$$

(NB: $\nabla(u(\bar{x}))$ not the same as $\nabla u(\bar{x})$! Think of it as ∇v where $v(x) = u(\bar{x})$. Likewise $\Delta(u(\bar{x})) = \Delta v(x) \neq \Delta u(\bar{x})$)

Recall from last week that, writing $|x|^{2-n}$ as r^{2-n}, $f(r) = r^{2-n}$

$$\Delta(|x|^{2-n}) = f''(r) + \frac{n-1}{r} f'(r)$$

$$= (2-n)(1-n) r^{-n} + \frac{(n-1)}{r}(2-n) r^{1-n} = 0.$$

Now calculate $\nabla(|x|^{2-n})$ (details like last week)

$$\frac{\partial}{\partial x_i}(x_1^2 + \dots + x_n^2)^{\frac{2-n}{2}} = (2-n)\, x_i\, (x_1^2 + \dots + x_n^2)^{\frac{2-n}{2}-1}$$

$$= \frac{(2-n)\, x_i}{|x|^n}$$

Now calculate ~~$\nabla(\bar{u})$ and $\Delta(\bar{u})$~~. $\nabla(u(\bar{x}))$ and $\Delta(\bar{u}(\bar{x}))$

(Writing $v(x) = u(\bar{x})$).

$$\frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i} u\left((x_1, \dots, x_n)/(x_1^2 + \dots + x_n)^2\right)$$

$$= \sum_{j=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_j} u_{x_j}(\bar{x})$$

$$\frac{\partial^2 v}{\partial x_i^2} = \sum_{j=1}^{n} \left(\left(\frac{x_i}{|x|^2}\right)_{x_j} u_{x_j}(\bar{x}) \right)_{x_i} \qquad a_{ij} \qquad a_{ik}$$

$$= \sum_{j=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_j x_i} u_{x_j}(\bar{x}) + \sum_{j=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_j} \sum_{k=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_k} u_{x_j x_k}(\bar{x})$$

$$\Delta(u(\bar{x})) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_j x_i} u_{x_j}(\bar{x}) + \underbrace{\left(\frac{x_i}{|x|^2}\right)_{x_j} \sum_{k=1}^{n} \left(\frac{x_i}{|x|^2}\right)_{x_k} u_{x_j x_k}(\bar{x})}_{A}$$

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} a_{ik}\, u_{x_j x_k}$$

$$= \langle (\nabla_x \bar{x})^t (\nabla_x \bar{x}), \nabla^2 u(\bar{x}) \rangle \quad \text{- dot product in } \mathbb{R}^{n \times n}$$

$$= |x|^{-4} \langle I, \nabla^2 u \rangle = |x|^{-4} \Delta u(\bar{x}) = 0.$$

Here we have

$$\text{(*)}\quad \Delta(\bar u(x)) = \overbrace{2\sum_{i=1}^n \left(\frac{(2-n)x_i}{|x|^{n}} \cdot \sum_{j=1}^n \left(\frac{x_i}{|x|^2}\right)_{x_j} u_{x_j}(\bar x) \right)}^{B}$$

$$+ \sum_{i=1}^n \sum_{j=1}^n \underbrace{\left(\frac{x_i}{|x|^2}\right)_{x_j x_i} u_{x_j}(\bar x)}_{c_{i,j}} \; |x|^{2-n}$$

Recall $\left(\frac{x_i}{|x|^2}\right)_{x_j} = \frac{\delta_{ij}}{|x|^2} - 2x_i x_j$

So $$B = \sum_{j=1}^n \underbrace{\left(\sum_{i=1}^n \frac{(2-n)x_i}{|x|^n} \left(\frac{\delta_{ij}}{|x|^2} - \frac{2x_ix_j}{|x|^4}\right)\right)} u_{x_j}(\bar x)$$

$$= \sum_{i=1}^n \frac{(2-n)x_i\delta_{ij}}{|x|^{n+2}} - \frac{2(2-n)x_i^2 x_j}{|x|^{n+4}}$$

$$= \frac{(2-n)x_j - 2(2-n)x_j}{|x|^{n+2}} = \frac{(n-2)x_j}{|x|^{n+2}}$$

ad C: $\left(\frac{x_i}{|x|}\right)_{x_i x_j} = \left(\frac{1}{|x|^2} - \frac{2x_i^2}{|x|^4}\right)_{x_j} = \frac{-2x_j}{|x|^4} + \frac{8x_i^2x_j}{|x|^6} - \frac{4x_i\delta_{ij}}{|x|^4}$

sum C_{ij} over i: $\sum_{i=1}^n \left(\frac{x_i}{|x|^2}\right)_{x_i x_j}$

$$= \frac{-2nx_j}{|x|^4} + \frac{8x_j}{|x|^4} - \frac{4x_j}{|x|^4} = \frac{2(2-n)}{|x|^4}.$$

Hence (*) becomes

$$\Delta(\bar u(x)) = \cancel{2}\sum_{j=1}^n \underbrace{\frac{2(n-2)x_j}{|x|^{n+2}} u_{x_j}(\bar x) + \frac{2(2-n)}{|x|^4} u_{x_j}(\bar x)\, |x|^{2-n}}_{=0}$$

$= \cancel{\sum_{j=1}^n}$ So $\bar u$ is harmonic.